A model for strategic decision-making: internationalization mode valuation using real-options.

Abstract

This paper proposed a model for strategic decision-making: internationalization mode valuation using real-options. Our model extends both literatures about real options and IB.

We try to answer the "how" and "when" questions related to the international growth of a firm's business. How do firms decide to either proceed with exports or foreign direct investment? And when should one or the other option be optimally exercised? Our model exposes best strategy for different values of ambiguity. We find that ambiguity increases the minimum size of the foreign market needed in order for the firm to internationalize (through exports or FDI). Export is chosen for foreign markets of small size and FDI for larger size because of the breakeven point, which is higher. But we also show when ambiguity is high, that the minimum size of the foreign market needs to be so large that the company will prefer to enter the foreign market directly through FDI. We additionally show that ambiguity increases the value of the waiting option to internationalize. Finally we take into account the fact that the relationship between ambiguity and the minimum quantity necessary to enter a foreign market is modified by the risk level, the correlation of the foreign market with the market portfolio, and the life-time of the project.

Keywords: ambiguity, export, FDI, internationalization, real option, waiting option

Code JEL: G11, Q40.

Introduction

Internationalization of the business and the firms is a key issue in nowadays' global economy; it is "crucial to economic recovery and to absorption of unemployment" (UNCTAD, 2014:19). But moderate to low expectations about the growth of matured markets in the next future may interrogate about the actual opportunity to invest abroad.

Domestic exploitation of a business offers corporations an option to operate abroad which in turn raises two critical and interconnected questions: how do firms decide to either proceed with exports or foreign direct investment? Assuming that their decision is profit related what strategy is the most profitable? And when should one or the other option be optimally exercised? The 'how' and the 'when' questions to internationalize the business of a firm are clearly interrelated because the optimal timing intimately depends on the chosen entry mode, while the most profitable one is contingent on the firm's agenda.

In this paper, we exploit recent findings in real option theory (ROT). We reconsider the internationalization decision determinants and clarify the way exports and foreign direct investment (FDI) interact. We also examine the most salient features of an international project (such as costs, profits, duration and so) and their influence on the manager's decision to go abroad[1].

The 'how' and the 'when' to internationalize the business questions have generated important debates in the international business area. Regarding the internationalization mode decision (answering the 'how' question), two alternatives are often opposed: exports and FDI[2]. On one side authors such as Shaver (2013) wonder whether the international business

[1] The real option approach is quite popular among scholars and practitioners for different reasons. Designing real option models forces users to identify and stress key determinants of the investment decision. Real option models can provide qualitative as well as quantitative insights about the 'how' and the 'when' questions.. More importantly, models can be adjusted for real life applications and calibrated to real data. A potential drawback of this approach is that a too parsimonious model may overlook important aspects of the problem under scrutiny. Consequently, this approach requires further improvements to verify the robustness of previous results and the absence of artifact.

[2] There are numerous numbers of alternative tactics for internationalization mode such as e.g. imports, international joint ventures (IJV), mergers and acquisitions. Various forms of collaboration between companies

community needs more entry modes studies, while on the other side Hennart & Slangen (2015) recently urged for more investigation. As for the timing issue (the 'when'), the now well-known Born Global approach (Knight & Cavusgil, 2005; Madsen & Servais, 1997) or international new venture (Oviatt & Phillips McDougall, 1994) classically challenged and extended the traditional views as it is the case for the Uppsala model (Johanson and Vahlne, 1977). More recent researches have used the ROT approach to determinate the optimal entry timing of a company into a foreign market (see for instance: Campa, 1993; Dixit, 1989; Rivoli & Salorio, 1996; Li & Rugman 2007). The foreign entry deferment provides a company with the option to wait for more relevant information before making critical decisions such which market to enter and how much to invest.

From a ROT perspective, the internationalization project must be described not only by relevant characteristics such as costs and implementation details, but also by the nature of risk and uncertainty involved by decisions.

We will first discuss costs associated with modes of internationalization. Costs may be either fixed or variable and may include additional sunk costs if the set-up is irreversible. Entry sunk costs may be splited into information, distribution, and learning costs (Baldwin and Krugman, 1989; Bugamelli & Infante 2002; Dixit, 1989; Roberts and Tybout 1997; Das et al. 2007). Exports have lower sunk costs and fixed costs than FDI but higher variable costs (Helpman et al. 2003; Conconi et al., 2014). As a comparison IJV is in an intermediate situation between exports and FDI. Following Conconi et al. (2014) for low and intermediate profitability on a foreign market the company will choose exports, but for higher levels of profitability it may find it preferable to pay the fixed cost of a FDI in order to reduce variable costs.

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are also possible, but the present paper will not discuss these other approaches. Interested readers may consult Hennart and Slangen (2015) for a recent survey on this debate.

In a second time we discuss the case when the decision to go international is already made. There are several strategies in order to implement the project to internationalize the business (see e.g. European Commission, 2010 for a six classes typology). The firm either enter the foreign market immediately or later (at the opportune time) and in one big move or step by step (Kogut, 1983). Gradually entering the target market, with ROT in mind, implies that growth options must be considered at all steps in the analysis (Kogut and Kulatilaka, 1994). It is worth mentioning that favouring a partial investment rather than a single immediate move or just a deferment has many advantages. Buckley, Casson and Gulamhussen (2002) explain for instance that partially investing creates a real option for the manager to develop the business further and later (if profitable). While he can gather more information so that he will be able to run the project more successfully than if it was simply deferred. Brouthers and Dikova (2010) show that a growth option is associated with every mode of foreign market entry and not only with exports or IJV modes[3].

In our third point we discuss how uncertainty[4] is a central concept in the real option approach (Buckley & Casson 1998). Different kinds of uncertainty or risk may be distinguished such as exogenous versus endogenous uncertainties, idiosyncratic versus systematic risks, etc. The kind of uncertainty we address in this paper is exogenous because the level of uncertainty remains unaffected by the firm's actions. We use the terminology of Roberts and Weitzman (1981). Especially there are no occasions for firms at stake to gradually learn something about the foreign market (i.e.. there is no growth option in our analysis) [5]. In that instance, it is common to refer to the classical ROT main result: an increase in uncertainty raises the

[3] It should be clarified that integrating all possible growth, switching between modes (Kouvelis, Axarloglou, and Sinha, 2001) and abandonment options in a ROT based analysis is a real challenge. Practically it could be an endless project; therefore we are in favor of parsimonious frameworks, which produce (at least theoretically) some analytical results.

[4] In the present paper, uncertainty and ambiguity are used interchangeably as synonyms.

[5] By contrast, endogenous uncertainty is affected by the decision maker's actions and it may be reduced, for instance, through learning (see also Cuypers and Martin, 2010, Fisch, 2008, Folta, 1998 for further discussions). Thus firms facing endogenous uncertainty have an incentive to reduce the delay before investing, as it is the sole way to start reducing the uncertainty. However, Chi (2000) explains that in the learning phase, endogenous uncertainty will remain and keep a growth option value alive, alike the exogenous uncertainty case.

value of waiting and thus decelerates investment (see McDonald and Siegel, 1986 and Dixit and Pindyck, 1994, for a standard presentation). Finally uncertainty may be also understood as a "Knightian" uncertainty that is a risk or a mix of both. In our paper's set up we will consider both concepts. In the context of internationalization we investigate a couple of connected issues that challenge the classical positive relationship between risk and the traditional ROT option to wait before internationalization. The first point is the difference between the idiosyncratic and the systematic risk, while the second point considers the difference between probabilistic uncertainty (i.e. the risk) and Knightian uncertainty since Knight (1921). A recent strand in the ROT literature indeed urges to reconsider the classical negative relation between risk and investment timing. Sarkar (2000, 2003), Lund (2005) and Wong (2007) stress for instance that changing the volatility alone (i.e. while letting all other structural parameters constant), as often done in ROT, may hurt traditional basic finance principles[6]. In most situations, the optimal investment trigger is a non-monotonic function of the project volatility and not a strictly increasing one as thought earlier[7].

In the international context, firms' managers and decision makers hardly know with certainty the events' distribution probability . Therefore it appears necessary to consider specific approaches able to deal both with the Knightian uncertainty (or ambiguity) and the traditional risk concept. In addition these approaches can disentangle ambiguity from risk aversion. Indeed while considering an internationalization strategy for a project, some firms

[6] The negative relation is actually only valid when the risk-adjusted expected return of the project value stays invariant to risk, given that we know from Standard Finance Theory that the risk-adjusted expected return of the project is linearly and positively related to risk when this later is systematic. So the traditional result holds when the project risk is purely idiosyncratic (as it does not imply any special compensation) or when the correlation between the project value and the market value nicely decreases when the risk increases (Lund, 2005).

[7] More precisely, Sarkar, 2000, 2003; Wong, 2007 show that a couple of effects interact when volatility goes up. First and as predicted by standard models, a "risk effect" pushes up the optimal investment trigger, enhances the value of the option to invest in the project and makes delaying more beneficial. But, second, a "return effect" increases the opportunity cost of waiting and makes consequently delaying more costly. This latter arises from the upward adjustment of the project expected return. Authors find that the return factor dominates for low levels of risk, while the risk factor dominates for high levels of risk.

might be averse to ambiguity, while other to risk or even both. So an appropriate decision-making model should account for both kinds of aversion.

To deal with ambiguity, we assume that the firm has a *set* of several probability measures over uncertain states rather than a single measure as traditionally used in the real option theory.In order to model Knightian uncertainty, we use the k -ignorance specification, following Chen and Epstein (2002) and Nishimura and Ozaki (2007)[8]. To capture firms' ambiguity aversion, we apply the multiple prior approach. It postulates that managers make decision on the basis of the worst-case scenario. In other words, considered preferences are related to the maxmin expected utility of Gilboa and Schmeidler (1989). Ambiguity is confined to the specific risk component. This is a key point to consider when designing Knightian uncertainty.

Finally, finite life foreign investment is the last characteristic we are considering. In a fast-changing global world, the expected time life of a given internationalization project is hypothesized to be shorter than the infinite life hypothesis in the traditional waiting option model. It is commonly known that the value of the project decreases when its lifetime decreases (see Gryglewicz et al., (2008) in a setting with no ambiguity). Therefore the option to defer is particularly valuable if the company is aiming to maintain its ownership advantages for the long run and if the investment is complexe to reverse.

This present paper uses recent findings of the real option literature. It explores whether and to which extent three dimensions can impact the timing and mode of foreign entry. These three dimensions are related to characteristics of foreign investments compared to the domestic one, and yet unexplored in the IB literature. They are: a) the Knightian uncertainty, b) the correlation of the foreign and domestic cash flows with the market portfolio, and c) the finite

[8] The k-ignorance model is one possible specification of the multiple prior approach. Other ways exist. Klibanoff et al. (2005) introduce, for instance, smooth preferences to model ambiguity aversion. This concept however has been shown difficult to apply in a continuous-time setting (Hansen and Sargent, 2010).

life of the project. This paper also contributes to the real option literature by gathering in a single framework, important issues that were so far considered separately.

The paper is organized as follows: section 1 delineates the model characteristics, the determinants of the foreign entry mode choice , the optimal internationalization trigger and the value of the option to wait. Section 2 offers numerical analyses to gain further insight of the specific case of foreign investment. Section 3 concludes.

1. Investing in Finite-Life foreign projects under Risk and Idiosyncratic Uncertainty

ROT recommends investing only when marginal benefits from investing exactly cover the sunk costs plus the value to undertake the project later. It is this possibility to defer the project that is valuated using option-pricing models. An option value is always positive so the value of the project must be significantly larger than the investment cost to start investment in contrast with the classical net present value (NPV) theory (Majd, and Pindyck 1987). Option value is a positive function of uncertainty in consequence the optimal timing recedes further when uncertainty increases. This classical ROT result serves as routine arguments in the IB literature to explain delays for investing abroad (Eden, 2009; Jiang, Aulakh, & Pan, 2009).

In standard real option models, a *single* probability measure is considered to assess future market conditions and the investment opportunity. But when managers face ambiguity, they rather think about a set of probability measures. If furthermore they are averse to ambiguity, it is often assumed that they make decisions under the worst-case scenario. They choose the probability distribution leading to the lowest outcome. Preferences considered here are based on the maxmin expected utility of Gilboa and Schmeidler (1989). Authors examining the role of ambiguity on the value of irreversible investment opportunity find that the effect of ambiguity dramatically differs from the effect of risk. An increase in ambiguity reduces the value of an investment opportunity (Nishimura and Ozaki, 2007; Trojanowska

and Kort, 2010) and ambiguity can delay (like risk) but accelerate investment decisions (Trojanowska and Kort, 2010; Miao and Wang, 2011). Miao and Wang (2011) show indeed that a decision to accelerate or postpone investments critically depends on whether ambiguity remains once investment is undertaken.

1.1. A framework for modelling product price under idiosyncratic ambiguity

Our real option framework builds on Sarkar (2000), Wong (2007), Nishimura and Ozaki (2007), Gryglewicz et al. (2008) and Trojanowska and Kort (2010) to account for correlation with the capital market, finite-life project, risk and ambiguity. In our setting, trading can take place continuously on foreign markets. There exists a risk free asset paying a constant interest rate denoted by r_f. Uncertainty in this environment is captured, in traditional ROT, by a benchmark probability measure used by firms to analyze investment projects supposing that there is no ambiguity.

Let $P_{m,t}$ represent the value of the capital market portfolio at time t. Then dynamics of the capital market portfolio are described by a geometric Brownian motion, which is defined by:

$$dP_{m,t} = \mu_m P_{m,t} dt + \sigma_m P_{m,t} dB_t^m . \tag{1}$$

μ_m and σ_m are two positive constant that represent the expected growth rate or drift of the market portfolio and its volatility respectively, and $dB_{m,t}$ is the increment of a standard Brownian motion. The real number $(\mu_m - r_f)/\sigma_m := \lambda$ stands for the market price of risk.

Let us neglect ambiguity for a while. Assuming that the dynamics of a product price on a foreign market are given by a geometric Brownian motion, as follows:

$$dP_t = \mu P_t \, dt + \sigma P_t \, dZ_t \tag{2}$$

where dZ_t stands for the increment of a standard Brownian motion that is correlated to random shocks affecting the market portfolio (dB_t^m). Denoting by $\rho \in [-1,1]$ the constant

instantaneous correlation between both, one has $dB_t^m dZ_t = \rho dt$ and uncertainty in the price process may be decomposed into two components: a systematic component related to the capital market and a specific one. More formally, equation (2) becomes

$$dP_t = P_t \left[\mu dt + \rho\sigma dB_t^m + \sqrt{1-\rho^2}\sigma dB_t^{\perp} \right] \tag{3}$$

where $B^{\perp}$ is a standard Brownian motion uncorrelated to B_m. Here the term $\rho\sigma dB_t^m$ simply represents the systematic component, while the term $\sqrt{1-\rho^2}\sigma dB_t^{\perp}$ captures the specific component.

From now on, one posits that ambiguity impacts only the specific component of the above dynamics, because the capital market portfolio concentrates information from a large number of different investors. Following Nishimura and Ozaki (2007) and Trojanowska and Kort (2010), we consider a positive real number k to represent Knightian uncertainty. Put it simple, the higher k, the higher the ambiguity on the specific component. In a different context, Chen and Epstein (2002) call this particular form of Knightian uncertainty "k-ignorance". Following the multiple prior (or minmax) approach, we posit that ambiguity aversion forces firms, managers and decision makers to consider the worst case scenario. In this case, it can be demonstrated that-uncertainty reduces the price growth rate and Equation (3) becomes

$$dP_t = P_t \left[\left(\mu - k\sqrt{1-\rho^2}\sigma \right) dt + \sigma \left(\rho dB_t^m + \sqrt{1-\rho^2} dB_t^{\perp,k} \right) \right] \tag{4}$$

where $B^{\perp,k}$ is a standard Brownian motion uncorrelated to B_m in a new probabilistic environment. The latter stochastic differential equation simply describes the dynamics of a geometric Brownian motion. Consequently, idiosyncratic ambiguity impacts the expected growth rate of the price as soon as correlation with the market portfolio is not perfect. The

above dynamics will serve as a basis for the valuation of internationalization modes and the associated option to wait.

1.2. Value of newly launched internationalization projects under idiosyncratic ambiguity

The valuation of an established internationalization project uses the classical discounted cash-flow (DCF) model. The objective is to explore the conditions under which the internationalization project will be financially viable. In order to simplify the presentation and notations, we present first the pricing of a generic internationalization project and will then specify the differences between the foreign projects. The actual value of an internationalization project depends on the internationalization mode chosen by the firm. For the theoretical models and associated simulations we will concentrate on two modes: export and FDI.

1.2.1. Valuation of a generic internationalization project

Consider a project that consists in selling a product abroad at price P (evolution of P is given by equation 4) for τ consecutive years. The project generates from inception time t_0 to expiration time $t_0 + \tau$ some random margins denoted by $\pi_t = Q_t\left(P_t - c\right)$. Here Q_t is the quantity sold at time t, P_t the prevailing price and c unit variable cost supposed to be constant over time. Using the standard DCF approach may then provide the present value of all expected future margins till expiration. Assuming constant quantities over the time life of the project, one has

$$W_{t_0} = Q\left\{\min_{[-k,k]} E\left[\int_{t_0}^{t_0+\tau} e^{-r(s-t_0)} P_s ds\right] - E\left[\int_{t_0}^{t_0+\tau} e^{-r_f(s-t_0)} c ds\right]\right\} \tag{5}$$

where the minimum operator reflects the firm's aversion towards ambiguity. Inspection of equation (5) reveals that both r and the risk free interest rates r_f are used to discount future cash flows. The discount rate r is associated to future random revenues that are correlated to

future prices of the capital market portfolio. Consequently, it can be evaluated to $r = r_f + \lambda\rho\sigma$ in our setting (cf. standard results of the capital asset pricing model (CAPM)). The risk free interest rate is used for future fixed costs because they are supposed constant through time. The present value of variable costs amounts to $\frac{c}{r_f}\left(1-e^{-r_f\tau}\right) \equiv c\,g(\tau, r_f)$. Computing the present value of future random revenues is, by comparison, less common in literature. By equation (4), the random revenue received at time t is

$P_t = P_{t_0} \exp\left[\left(\mu - k\sqrt{1-\rho^2}\sigma - \frac{1}{2}\sigma^2\right)(t-t_0) + \sigma\left(\rho\left(B_t^m - B_{t_0}^m\right) + \sqrt{1-\rho^2}\left(\tilde{B}_t^{\perp,k} - \tilde{B}_{t_0}^{\perp,k}\right)\right)\right]$ and the corresponding expectation at time t_0 is $E[P_t] = P_{t_0} \exp\left[\left(\mu - k\sqrt{1-\rho^2}\sigma\right)(t-t_0)\right]$. To go further in computations, it is important to stress that the difference between r (the rate used to discount future revenues) and $\mu - k\sqrt{1-\rho^2}\sigma$ (the expected rate of growth of product prices) is the convenience yield of the project denoted by $\delta(k,\rho,\sigma)$[9]. Because $r = r_f + \lambda\rho\sigma$, one may write $\delta(k,\rho,\sigma) := r - \left(\mu - k\sqrt{1-\rho^2}\sigma\right) = r_{\sigma,k,\rho} - \mu$ where $r_{\sigma,k,\rho} = r_f + \sigma\left[\lambda\rho + k\sqrt{1-\rho^2}\right]$. This expression then shows that Knightian uncertainty is just an additional source of compensation and, by passing, that the parameter for ambiguity aversion (k) is also a measure of the reward for Knightian uncertainty (expressed in unit of volatility). We can also observe that the correlation coefficient impacts the convenience yield (and hence the decision making) through a couple of different channels. The systematic component ($\sigma\lambda\rho$) is linear w.r.t. the correlation coefficient and it depends on the sign of this correlation. Quite differently, the ambiguity component ($\sigma k\sqrt{1-\rho^2}$) is non linear and does

[9] For infinite-life projects (those having $\tau = \infty$) and no ambiguity, it is standard to assume $r > \mu_i$ to ensure convergence of the expected present value of future revenues. Obviously this condition is not strictly necessary for finite-life projects, because convergence is warranted by construction. But, for sake of comparability and reasons related to the option to invest, one will however assume that this inequality holds.

not depend on the correlation sign. Finally, the present value of revenues equals (proof in appendix):

$$\min_{\theta\in[-k,k]} E\left[\int_{t_0}^{t_0+\tau} e^{-r(s-t_0)} P_s ds\right] = P_{t_0} \frac{1-e^{-\left(r_f+\sigma\left[\lambda\rho+k\sqrt{1-\rho^2}\right]-\mu\right)\tau}}{r_f+\sigma\left[\lambda\rho+k\sqrt{1-\rho^2}\right]-\mu} \equiv P_{t_0} g(\tau,\delta(k,\rho,\sigma))$$

This value critically depends on the convenience yield $\delta(k,\rho,\sigma)$; it is therefore a function of the risk, the ambiguity, the correlation with capital market and of course the time life of the project. Combining the present value of revenues and the present value of costs, we obtain a closed-form expression of the project present value:

$$W_{t_0} = Q\left[P_{t_0} g(\tau,\delta(k,\rho,\sigma)) - cg(\tau,r_f)\right] \tag{6}$$

Equation (6) highlights how structural parameters may impact the firm's perspective at the inception of the project independently from fixed and investment costs. For some structural parameters, this present value of future revenues net of variable costs can be negative. In such cases, the project is simply not worth undertaking. This is an important aspect for simulations as it may restrict the range of parameters we consider hereafter. It is also worth mentioning that the valuation formula (6) extends Sarkar (2000), Wong (2007), Nishimura and Ozaki (2007), Gryglewicz et al. (2008) and Trojanowska and Kort (2010) in different aspects. Wong (2007) assumes the value of the project as given. Neither Sarkar (2000) nor Gryglewicz et al. (2008) introduce ambiguity in their investment opportunity. Plugging $k=0$ into equation (6) exactly provides result of Gryglewicz et al. (2008) and Sarkar (2000) respectively. For their parts, Nishimura and Ozaki (2007) and Trojanowska and Kort (2010) do not consider any idiosyncratic component in the random revenue. Their ambiguity rather concerns the systematic risk of the project.

Now we hypothesized that, to operate the project, a company has to pay a sunk cost I and they are constant (and continuous) fixed costs f during the life of the project. The

present value of these fixed costs amounts to $F(\tau)=E\left[\int_{t_0}^{t_0+\tau} e^{-r_f(s-t_0)} f ds\right]=\frac{f}{r_f}\left(1-e^{-r_f\tau}\right)= f\, g(\tau,r_f)$ at the inception time t_0. Finally, the overall NPV of an established project at its inception time t_0 is equal to

$$NPV_{t_0} = Q\left[P_{t_0} g(\tau,\delta(k,\rho,\sigma)) - cg(\tau,r_f)\right] - f\, g(\tau,r_f) - I \tag{7}$$

which is a linear function of the sold quantity Q which is a proxy for the foreign market size,

1.2.2. Valuation of export and FDI projects

Characteristics of the generic project are now specified to take into account differences between foreign entry modes either export or FDI. Following the previous literature, the specific characteristics of each entry modes are presented in table 1.

Table 1 Cost structure of entry modes

Entry modes	Exports (E)	FDI
Investment (I)	Low	High
Variable costs (c)	High	Low
Fixed cost(f)	Low	High

By using equation (7) with these specifications, NPVs of export and FDI are respectively:

$$NPV(E) = Q\left[Pg(\tau,\delta(k,\rho,\sigma)) - c_E g(\tau,r_f)\right] - f_E g(\tau,r_f) - I_E$$

and

$$NPV(FDI) = Q\left[Pg(\tau,\delta(k,\rho,\sigma)) - c_{FDI} g(\tau,r_f)\right] - f_{FDI} g(\tau,r_f) - I_{FDI}\,.$$

These two expressions differ only by the magnitudes of variable costs ($c_{FDI} < c_E$), fixed costs ($f_E < f_{FDI}$) and sunk costs ($I_E < I_{FDI}$) as described in table 1. We can derive for all

strategies the minimal quantity, or breakeven point, that makes a given foreign entry mode profitable. For example, export is a positive NPV project if the quantity is larger than

$$Q_E^{Min} = \frac{f_E g(\tau, r_f) + I_E}{Pg(\tau, \delta(k, \rho, \sigma)) - c_E g(\tau, r_f)} \tag{8}$$

Q_E^{Min} represents the break-even point in order for the company to export. The size of the foreign market must be large enough in order to cover the sunk cost and the present value of fixed costs. Note that the higher the sale price the lower the minimum market size.

Taking into account the cost structures of the two projects (export and FDI), it is easy to show that there is a quantity Q_{EFDI} such that the company switch between export and FDI that is $\begin{matrix} if\ Q \le Q_{EFDI} \Rightarrow NPV(E) \ge NPV(FDI) \\ if\ Q > Q_{EFDI} \Rightarrow NPV(E) < NPV(FDI) \end{matrix}$. Or, by replacing NPVs by their expressions,

$$Q_{EFDI} = \frac{(f_{FDI} - f_E) g(\tau, r_f) + I_{FDI} - I_E}{(c_E - c_{FDI}) g(\tau, r_f)} = \frac{(f_{FDI} - f_E)}{(c_E - c_{FDI})} + \frac{I_{FDI} - I_E}{(c_E - c_{FDI}) g(\tau, r_f)} \tag{9}$$

The firm will prefer to enter the foreign market using directly the FDI entry mode if $Q_E^{Min} \ge Q_{EFDI}$. Our ultimate objective is to characterize the optimal condition to define the t_0 point in which to launch the project.

1.3. Valuation of the option to internationalize under idiosyncratic ambiguity

At any time, the firm can decide to launch the foreign project by paying the investment sunk cost I so as to receive a continuous profit flow for a τ-long period. In order to derive analytical solutions, we assume that the decision to enter a foreign market may be postponed constantly, so that the option to invest, whose value is denoted by V, is similar to a perpetual American call option written on the project value with strike I.

As shown by predecessors, the optimal stopping time approach still applies in a context with ambiguity and firm's aversion to it. According to the usual first passage time criterion, the manager must invest when the first time W reaches a deterministic exercise threshold. Because the option to invest has no expiration, the exercise threshold is just a constant real number W^* and the value of the option to invest is just a function of W and not a function of time, hence $V_t = V(W_t)$. When $W = W^*$, the firm is perfectly indifferent whether to invest in the project and keep the option open $V(W^*) = NPV(W^*)$.

An ambiguity-averse firm will exercises optimally its option to invest in an internationalization project of time-life τ as soon as the project value process W reaches the time-life dependent threshold (demonstration in appendix)

$$W^*(\tau) = \frac{\beta}{\beta - 1}\left(I + fg(\tau, r_f)\right) \tag{10}$$

where β is the positive root of the quadratic function $\frac{1}{2}\sigma^2\beta(\beta-1) + m\beta - r_f = 0$ with m given in appendix. More explicitly, β is given by

$$\beta = \frac{1}{2} - \frac{\mu - \left(k\sqrt{1-\rho^2} + \lambda\rho\right)\sigma}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu - \left(k\sqrt{1-\rho^2} + \lambda\rho\right)\sigma}{\sigma^2}\right)^2 + \frac{2r_f}{\sigma^2}}.$$

It is larger than one and so is the term $\frac{\beta}{\beta - 1}$ often called the "hysteresis factor". The corresponding value of the option to invest in a foreign project is then

$$V(W_t) = \left(W^*(\tau) - fg(\tau, r_f) - I\right)\left(\frac{W_t}{W^*(\tau)}\right)^{\beta} \tag{11}$$

for all (admissible i.e. positive) values W_t lower than the threshold $W^*(\tau)$. In other words, the firm has an incentive to invest when the project value is strictly greater than the investment added to the present value of fixed costs. This result contradicts the standard

conclusion of the NPV criterion. The formula (11) is straightforward to understand because $\left(W^*(\tau) - fg(\tau, r_f) - I\right)$ represents the value of the project at inception and $\left(\frac{W_t}{W^*(\tau)}\right)^{\beta}$ stands for the present value of one unit of money received when the project is launched. By inspecting both the value at inception and the discounting factor, it is clear that the time-life of the project plays a very subtle role. As the time-life of the project increases, the present value of total fixed costs at inception increases but the value of the project at inception enlarges (up to a limit equals to $\frac{\beta}{\beta-1}\left(I + \frac{f}{r_f}\right)$) and the discounting factor decreases. The rise of the decision threshold $W^*(\tau)$ postpones the inception of the project.

The foreign investment threshold can also be expressed as a function of the price of the product sold in the foreign market. Replacing equation (10) $W^*(\tau)$ by its expression given by equation (6) we obtain $Q\left[P_E^* g(\tau, \delta(k, \rho, \sigma)) - c_E g(\tau, r_f)\right] = \frac{\beta}{\beta-1} \frac{I_E + f_E g(\tau, r_f)}{g(\tau, \delta(k, \rho, \sigma))}$.

We have already demonstrated that exports mode is profitable for a minimum market size Q_E^{Min}. Therefore replacing Q by Q_E^{Min} (given by equation (8) in the previous equation, the exporting price threshold, P_E^*, is solution of

$$P_E^* g(\tau, \delta(k, \rho, \sigma)) - c_E g(\tau, r_f) = \frac{\beta}{\beta-1}\left[P_E^{Min} g(\tau, \delta(k, \rho, \sigma)) - c_E g(\tau, r_f)\right] \qquad (12)$$

P_E^{Min} : price such as the NPV of an export project of a given market size equals 0.

This equation tells us that company will export only if the present value of revenues over variable costs is equal to the hysteresis factor times the present value of margin over variable cost, such as the NPV of exports equals 0 (classical NPV criterion).

Knowing that company will use the FDI entry mode if $Q_E^{Min} = Q_{E/FDI}$, (equation (9)) the price threshold such as the FDI is chosen as entry mode is given by

$$P_{FDI}^{*} g(\tau,\delta(k,\rho,\sigma)) - c_{FDI} g(\tau, r_f) = \frac{\beta}{\beta-1} \frac{f_{FDI} g(\tau, r_f) + I_{FDI}}{f_E g(\tau, r_f) + I_E} \left[P_E^{Min} g(\tau,\delta(k,\rho,\sigma)) - c_E g(\tau, r_f) \right]$$

And, using equation (12) $\frac{P_{FDI}^{*} g(\tau,\delta(k,\rho,\sigma)) - c_{FDI} g(\tau, r_f)}{P_E^{*} g(\tau,\delta(k,\rho,\sigma)) - c_E g(\tau, r_f)} = \frac{f_{FDI} g(\tau, r_f) + I_{FDI}}{f_E g(\tau, r_f) + I_E}$

The FDI price threshold is such as the thresholds margin ratio equals the cost margin ratio of the two entry modes. As the fixed and sunk costs of FDI are greater than those of exports, the FDI price threshold is necessarily higher than the export price threshold. And, taking into account the fact that the cost ratio is quite simple we will concentrate our simulation analysis on the behavior of the export price threshold.

2. Numerical simulations

As it is emphasized by Li (2007), in a critical review of applications of real option theory to IB, ROT allows to identify rigorously the evolution of uncertainty and to specify the relationship between parameters of interest (here, the size of the market, Q, the unit price of the product, P, the expected growth of the price μ, classical and Knightian uncertainty, σ and k, the correlation of price evolutions with the market portfolio ρ, the variable, fixed and sunk costs, respectively c, f, I, and the life time of the project τ) and the valuation of real options in international investment. He also states that option pricing models are particularly useful when real-life data are absent. In these cases, using simulation techniques can lead to meaningful empirical results. In our case, data could be available for some sectors, companies and products, but the objective of the paper is not to compute a precise value for the various thresholds of a specific internationalization case but rather to explore the functional relationship between independent variable and the projects internationalization NPV and option values. We will focus on the specific determinants introduced in the option

analysis i.e. the ambiguity level, correlation with the market and time-life of the project. We therefore rely on simulations to illustrate salient features of our setting.

We will first investigate the impact of ambiguity, correlation with the capital market, and time-life of the project on the NPV of the two foreign entry modes and then on the threshold price to export using the following baseline scenario:

Base case parameters:

rf =0.03; Σ=0.2; λ=0.3; ρ=0.4; κ=0; τ=15; fc = 1.5; IC=100.; μ=rf+λ Σ ρ;

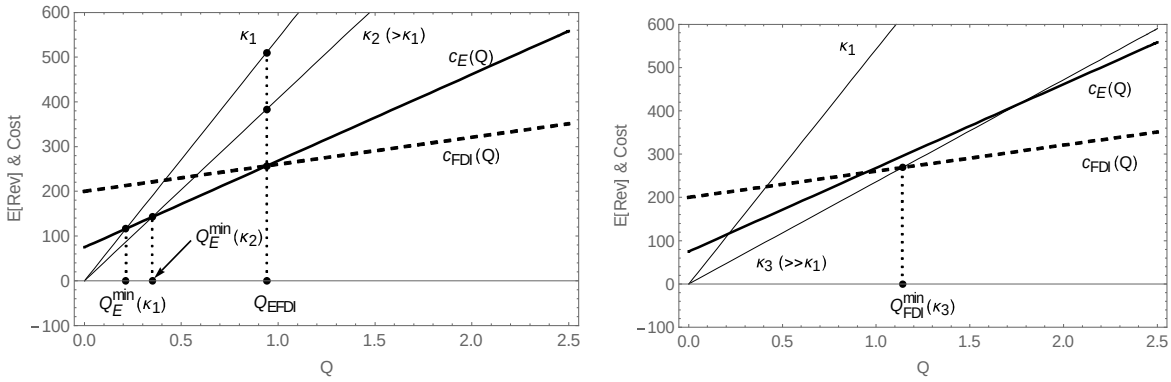
mu[s_]:=rf+λ s ρ; τ=15;P=50; cE=16; cFDI=5; fE=0; fFDI=1.1 fE; IE=75; IFDI=200; κ=.7;

Obviously, nothing prevents us to consider other parameters values to match the present analysis with a more specific situation.

2.1. NPV Analysis of Internationalization Modes

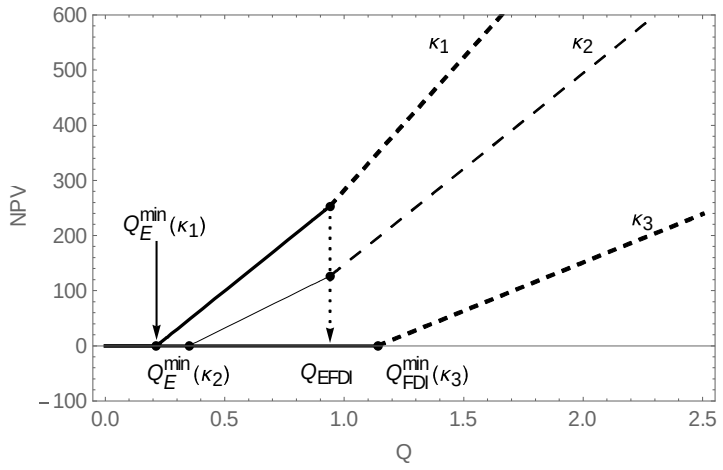
We first investigate the impact of ambiguity level on the minimum size of the market necessary for the firm to internationalize (figure 1). As ambiguity reduces the present value of the firm's revenue but has no effect on the costs, the minimum size of the market Q_E^{Min} must increase when ambiguity goes up, as illustrated by figure 1 and 2.

Figure 1: Expected revenue and costs.



In the left graph, the ambiguity level is low or moderate and the expected revenue that depends on these ambiguity levels (in thin lines) get greater than the variable cost of exporting when the quantity becomes larger than $Q_E^{Min}(k_1)$ and $Q_E^{Min}(k_2)$ respectively. When the quantity reaches Q_{EFDI}, the variable cost of exporting is larger than the one of investing so the company has an incentive to switch to FDI. In the right graph, we plot a high ambiguity level in addition to the benchmark case. The break even point of exports mode is reached for a larger market size than for the FDI mode. The ambiguity is so high here that the size of the market must be large enough to obtain a positive NPV (figure 2). In this case, there is no export decision before proceeding to FDI. Following ROT, FDI entry mode should thus be observed for large and highly ambiguous foreign markets.

Figure 2: Net present Values of the best strategy for different values of ambiguity.

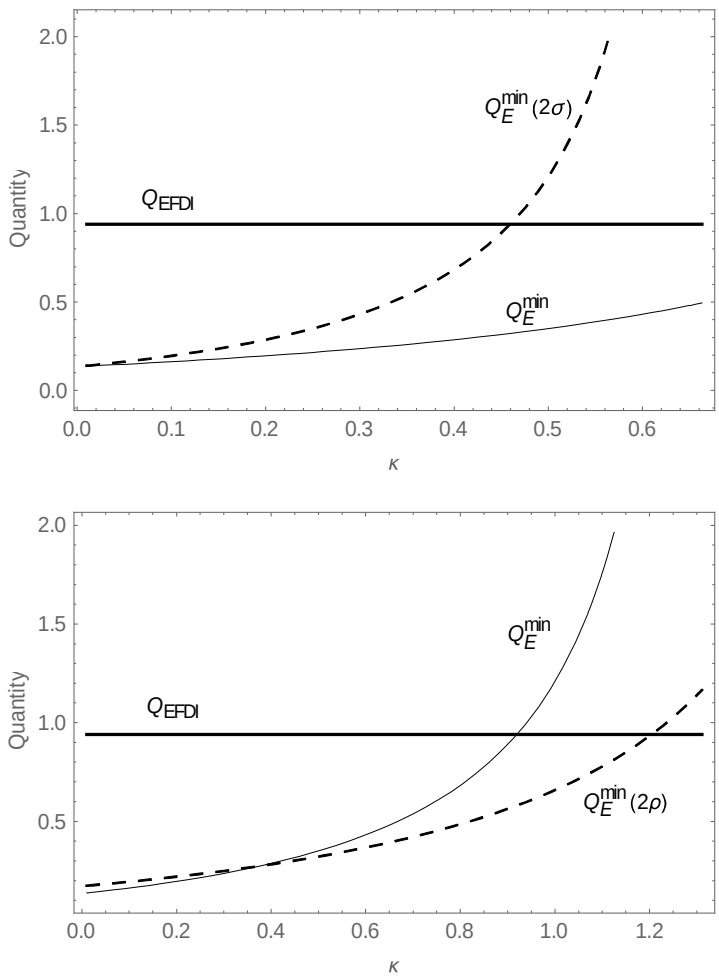


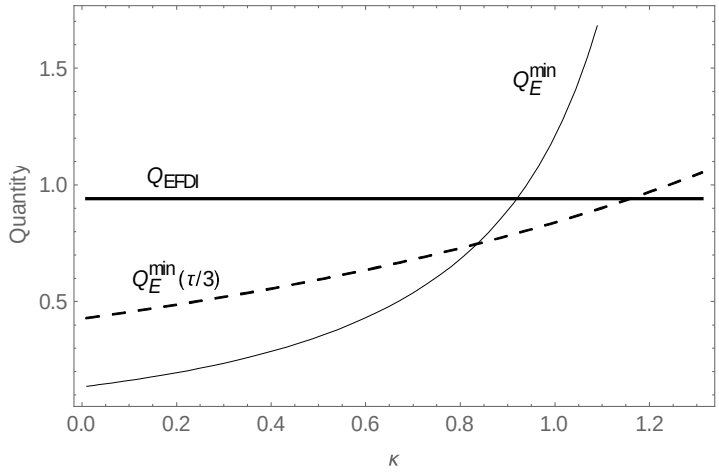
Under k1 (low ambiguity), a firm should avoid business internationalization when sales are too low (lower than $Q_E^{Min}(k_1)$), above that threshold they can export and turn to FDI for quantities above Q_{EFDI}. Same for intermediate ambiguity except that the NPV is lower. Finally, when facing significant ambiguity level, we directly turn to FDI solutions without experimenting export.

In figure 3-a we observe that the slope of the relationship between ambiguity and the minimum market size increases dramatically with the level of risk ($Q_E^{Min}(2\sigma)$ meaning that the level of risk doubled compared to the baseline level). For high risk and ambiguous markets, the minimum market size must be much higher than without ambiguity in order to attract foreign companies, illustrating the necessity to take the ambiguity effect into account. Moreover, on this kind of foreign markets FDI entry mode should be observed more frequently than it is expected when using classical ROT. Correlation with the market portfolio has a complex effect (figure 3-b). When ambiguity is small, an increase of correlation advances (slightly) the breakeven point but it is the contrary under large ambiguity. One must remember that when correlation increases, the weight of systematic risk rises too, whereas the weight of specific risk drops. As ambiguity only affects the specific

risk, its negative impact on revenue is therefore reduced. Finally (figure 3-c), the slope of the relationship between ambiguity and the minimum market size decreases when the lifetime of the project goes down ($Q_E^{Min}(\tau/3)$ meaning that the lifetime has been divided by three compared to the baseline level). So for low ambiguity, shorter project will increase the breakeven point and it is the reverse causality for high ambiguity.

Figure 3: Breakeven quantities for different level of structural parameters (risk, correlation and time-life length), Q_E^{Min} is the benchmark case.

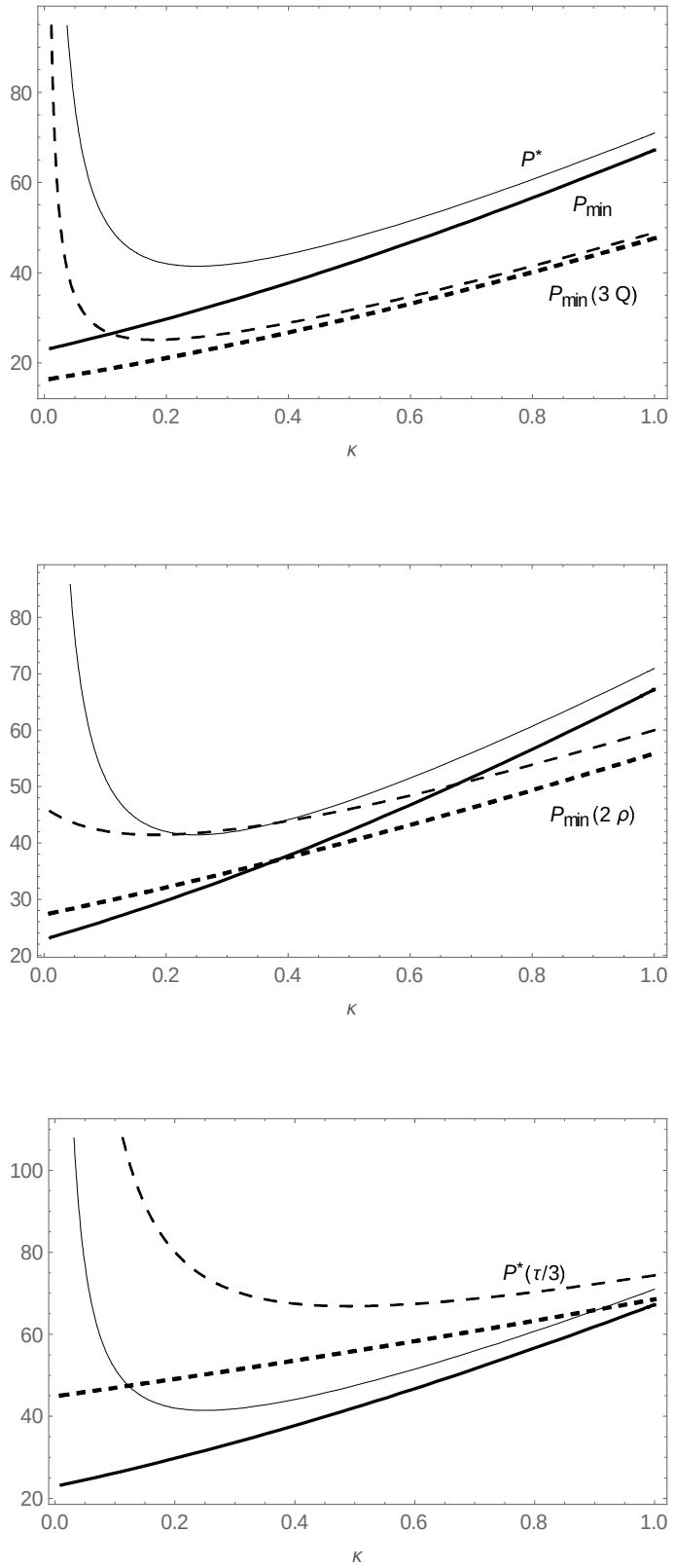




2.2. Delay of the internationalization decision: 'option to wait'

Ambiguity impact on the breakeven price (price such as NPV of exports equals 0) and export price threshold is presented in figure 4. The export price threshold decreases dramatically for small and intermediate level of ambiguity (figure 4a). This effect is robust to market size changes (figure 4a when the size of the market is tripled), the correlation coefficient (figure 4b) and the time life of the project. Therefore, ambiguity and risk have very different impact on the 'option to wait' threshold.

Figure 4: Impact of ambiguity on breakeven end threshold Prices of export



Conclusion

This paper proposed a model for strategic decision-making: internationalization mode valuation using real-options. Our model extends both literatures about real options and IB.

We try to answer the “how” and “when” questions related to the international growth of a firm’s business. How do firms decide to either proceed with exports or foreign direct investment? And when should one or the other option be optimally exercised? Our model exposes best strategy for different values of ambiguity; we accounted for both cases of classic and Knightian uncertainty. We introduce ambiguity in our valuation formula and extend ROT in different aspects.

We find that ambiguity increases the minimum size of the foreign market (or the minimum foreign price) needed in order for the firm to internationalize (through exports or FDI). Export is chosen for foreign markets of small size and FDI for larger size because of the breakeven point, which is higher. But we also show when ambiguity is high, that the minimum size of the foreign market needs to be so large that the company will prefer to enter the foreign market directly through FDI. We additionally show that ambiguity increases the value of the waiting option to internationalize, i.e. the minimum quantity that triggers the internationalization decision increases with ambiguity. Finally we take into account the fact that the relationship between ambiguity and the minimum quantity necessary to enter a foreign market is modified by the risk level, the correlation of the foreign market with the market portfolio, and the life-time of the project.

Our model has the limitation to not be dynamic, indeed we do not account for flexible cases when for instance a firm would be willing to invest abroad after first exporting. There is no growth option in our analysis. We also exclude the possibility for the firm to intervene on the level of uncertainty. Those limitations should be integrated in further development of the real options applications in IB.

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Appendix

This appendix contains lengthy or non-straightforward proofs as well as some materials necessary for demonstrating results in the core text.

The first lemma concerns the random revenue within $\left(\Omega, \mathsf{F}, \{\mathsf{F}_t\}_t, \mathsf{Q}^\theta\right)$. This result is necessary for following demonstrations.

Lemma A: *Random revenue at time t amounts to*

$$\pi_t = \pi_{t_0} \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)(t - t_0) - \int_{t_0}^{t} \sqrt{1-\rho^2}\,\sigma\theta_u du + \sigma\left(\rho\left(B_t^m - B_{t_0}^m\right) + \int_{t_0}^{t} \sqrt{1-\rho^2}\, d\tilde{B}_u^{\perp,\theta}\right)\right]$$
$$= \pi_{t_0} \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)(t - t_0) - \int_{t_0}^{t} \sqrt{1-\rho^2}\,\sigma\theta_u du + \sigma\left(\rho\left(B_t^m - B_{t_0}^m\right) + \sqrt{1-\rho^2}\left(\tilde{B}_t^{\perp,\theta} - \tilde{B}_{t_0}^{\perp,\theta}\right)\right)\right]$$

in $\left(\Omega, \mathsf{F}, \{\mathsf{F}_t\}_t, \mathsf{Q}^\theta\right)$ *and the corresponding time* t_0*-conditional expectation is*

$$E^{\mathsf{Q}^\theta}\left[\pi_t \middle| \mathsf{F}_{t_0}\right] = \pi_{t_0} \exp\left[\left(\mu - \int_{t_0}^{t} \sqrt{1-\rho^2}\,\sigma\theta_u du\right)\right].$$

Nishimura and Ozaki (2007) prove a very useful general inequality (see their appendix A) we can reconsider in the context of $K = [-k; k]$.

Lemma B: For any $s \geq t$, *and for any* $\theta \in K$, *it holds that*

$$E^{\mathsf{Q}^\theta}\left[\exp\left[-\int_t^s \sigma\theta_u du + \sigma\left(\tilde{B}_s^{\perp,\theta} - \tilde{B}_t^{\perp,\theta}\right)\right] \middle| \mathsf{F}_t\right] \geq E^{\mathsf{Q}^k}\left[\exp\left[-k(s-t) + \sigma\left(\tilde{B}_s^{\perp,k} - \tilde{B}_t^{\perp,k}\right)\right] \middle| \mathsf{F}_t\right].$$

The value of the generic project is equal to

$$W_{t_0} = \min_{\theta\in[-k,k]} E^{\mathbb{Q}^\theta}\left[\int_{t_0}^{t_0+\tau} e^{-r(s-t_0)}\pi_s ds \middle| \mathbb{F}_{t_0}\right]$$

$$\overset{Fubini}{=} \min_{\theta\in[-k,k]} \int_{t_0}^{t_0+\tau} E^{\mathbb{Q}^\theta}\left[e^{-r(s-t_0)}\pi_s \middle| \mathbb{F}_{t_0}\right] ds$$

$$\overset{lemma\ A}{=} \min_{\theta\in[-k,k]} \int_{t_0}^{t_0+\tau} \pi_{t_0} e^{\left(-r+\mu-\frac{1}{2}\sigma^2\right)(s-t_0)} E^{\mathbb{Q}^\theta}\left[e^{-\int_{t_0}^{s}\sqrt{1-\rho^2}\sigma\theta_u du+\sigma\left(\rho\left(B_s^m-B_{t_0}^m\right)+\sqrt{1-\rho^2}\sigma\left(\tilde{B}_s^{\perp,\theta}-\tilde{B}_{t_0}^{\perp,\theta}\right)\right)} \middle| \mathbb{F}_{t_0}\right] ds$$

$$\overset{lemma\ B}{=} \pi_{t_0}\int_{t_0}^{t_0+\tau} e^{\left(-r+\mu-\frac{1}{2}\sigma^2\right)(s-t_0)} E^{\mathbb{Q}^k}\left[e^{-\sqrt{1-\rho^2}\sigma k(s-t_0)+\sigma\left(\rho\left(B_s^m-B_{t_0}^m\right)+\sqrt{1-\rho^2}\sigma\left(\tilde{B}_s^{\perp,k}-\tilde{B}_{t_0}^{\perp,k}\right)\right)} \middle| \mathbb{F}_{t_0}\right] ds$$

$$= \pi_{t_0}\int_{t_0}^{t_0+\tau} e^{\left(-r+\mu-k\sqrt{1-\rho^2}\sigma\right)(s-t_0)} ds$$

and the expression given in equation (6) in the text follows by simple algebra.

The value of the option to invest (under idiosyncratic ambiguity) may then be written

$$\begin{aligned} V_t &= \max_{t'\geq t}\min_{\theta\in[-k,k]} E\left[\left(\int_{t'}^{t'+\tau} e^{-r(s-t)}\pi_s ds - e^{-r(t'-t)}I - e^{-r(t'-t)}F\right)\middle| \mathbb{F}_t\right] \\ &= \max_{t'\geq t}\min_{\theta\in[-k,k]} E^{\mathbb{Q}^\theta}\left[e^{-r(t'-t)}\left(\int_{t'}^{t'+\tau} e^{-r(s-t')}\pi_s ds - I - F\right)\middle| \mathbb{F}_t\right] \end{aligned}$$

$F(\tau)$ stands for the future total fixed costs generated by the project of time-life τ.

Consider a portfolio made of a long position in the option to invest and a short position in n units of the market portfolio. Its value at time t is given by: $\Pi_t = V_t - nP_{mt}$. The total return from holding the portfolio over an infinitesimal period of time dt, is

$d\Pi_t = dV_t - ndP_{mt} = \frac{\partial V}{\partial W}dW + \frac{1}{2}\frac{\partial^2 V}{\partial W^2}(dW)^2 - n\mu_m P_m dt - n\sigma_m P_m dB_{m,t}$ when we use the Ito's lemma to find the second equality. Stochastic calculus then yield to:

$$\begin{aligned} d\Pi_t = &\left[\frac{\partial V}{\partial W}\left(\mu-\theta_t\sqrt{1-\rho^2}\sigma\right)W_t + \frac{1}{2}\frac{\partial^2 V}{\partial W^2}(\sigma W_t)^2 - n\mu_m P_m\right]dt \\ &+\left[\frac{\partial V}{\partial W}W\rho\sigma - n\sigma_m P_m\right]dB_{m,t} \\ &+\left[\frac{\partial V}{\partial W}W\sqrt{1-\rho^2}\sigma\right]dB_t^{\perp,\theta} \end{aligned}$$

This equation reveals that, if one sets $n^* P_m = \frac{\partial V}{\partial W} \rho \frac{\sigma}{\sigma_m} W$, exposition to (non ambiguous) systematic risk disappears and the expected rate of return of the portfolio should be equal to the riskless rate (because remaining specific risk is diversifiable). Hence, in the continuation region, one has: $E^{Q^\theta}\left[d\Pi_t \middle| \mathcal{F}_t\right] = r_f \Pi_t dt$ so that:

$$\left(\frac{\partial V}{\partial W} W_t \left(\mu - \theta_t \sqrt{1-\rho^2}\sigma \right) + \tfrac{1}{2} \frac{\partial^2 V}{\partial W^2} W_t^2 \sigma^2 - n^* P_{m,t} \mu_m \right) dt = \left(V_t - n^* P_{m,t} \right) r_f dt .$$

We can then simplify by dt, plug n^* into this equation and rearrange terms. Finally, one obtains:

$$\tfrac{1}{2} \frac{\partial^2 V}{\partial W^2} W_t^2 \sigma^2 + \frac{\partial V}{\partial W} W_t \left(\mu - \theta_t \sqrt{1-\rho^2}\sigma + \frac{r - \mu_m}{\sigma_m} \rho\sigma \right) = r_f V_t \qquad \text{(A1)}$$

where we recognize $\frac{r - \mu_m}{\sigma_m} = -\lambda$.

The general solution of Eq. (A1) is $V(W) = AW^\beta + BW^\alpha$, with β (respectively α) the positive root of the quadratic equation: $\tfrac{1}{2}\sigma^2 x(x-1) + mx - r_f = 0$.

When the value of the project tends to zero, the option to invest becomes worthless $\lim_{W \to 0} V(W) = 0$. So B is necessarily equal to zero. The value matching condition forces the option to invest to be equal to its intrinsic value at exercise ($V(W^*) = W^* - F(\tau) - I = AW^{*\beta}$) leading to $A = \left[W^* - F(\tau) - I\right](W^*)^{-\beta}$.

The smooth pasting condition at W^* $\lim_{W \to W^*} V'(W) = 1$ provides W^*. One has: $\frac{dV(W)}{dW} = \left[\left[W^* - F(\tau) - I\right](W^*)^{-\beta}\right]\beta W^{\beta-1}$ and because $\left.\frac{dV(W)}{dW}\right|_{W=W^*} = 1$ we find

$$\left(W^* - F(\tau) - I\right)\beta\left(W^*\right)^{-1} = 1 \Leftrightarrow W^* = \frac{\beta}{\beta - 1}\left(F(\tau) + I\right) = W^*(\tau)$$

That is the expression given by equation (10) in the text.